

On the Modes of Polynomials Derived from Nondecreasing Sequences

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Abstract. Wang and Yeh proved that if $P(x)$ is a polynomial with non-negative and nondecreasing coefficients, then $P(x+d)$ is unimodal for any $d > 0$. A mode of a unimodal polynomial $f(x) = a_0 + a_1x + \cdots + a_mx^m$ is an index k such that a_k is the maximum coefficient. Suppose that $M_*(P, d)$ is the smallest mode of $P(x+d)$, and $M^*(P, d)$ the greatest mode. Wang and Yeh conjectured that if $d_2 > d_1 > 0$, then $M_*(P, d_1) \geq M_*(P, d_2)$ and $M^*(P, d_1) \geq M^*(P, d_2)$. We give a proof of this conjecture.

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1 Introduction

This paper is concerned with the modes of unimodal polynomials constructed from nonnegative and nondecreasing sequences. Recall that a sequence $\{a_i\}_{0 \leq i \leq m}$ is unimodal if there exists an index $0 \leq k \leq m$ such that

$$a_0 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_m.$$

Such an index k is called a mode of the sequence. Note that a mode of a sequence may not be unique. It is said to be spiral if

$$a_m \leq a_0 \leq a_{m-1} \leq a_1 \leq \cdots \leq a_{\lfloor \frac{m}{2} \rfloor}, \quad (1.1)$$

where $\lfloor \frac{m}{2} \rfloor$ stands for the greatest integer less than $\frac{m}{2}$. Clearly, the spiral property implies unimodality. We say that a sequence $\{a_i\}_{0 \leq i \leq m}$ is log-concave if for $1 \leq k \leq m-1$,

$$a_k^2 \geq a_{k+1}a_{k-1},$$

and it is ratio monotone if

$$\frac{a_m}{a_0} \leq \frac{a_{m-1}}{a_1} \leq \dots \leq \frac{a_{m-i}}{a_i} \leq \dots \leq \frac{a_{m-\lfloor \frac{m-1}{2} \rfloor}}{a_{\lfloor \frac{m-1}{2} \rfloor}} \leq 1 \quad (1.2)$$

and

$$\frac{a_0}{a_{m-1}} \leq \frac{a_1}{a_{m-2}} \leq \dots \leq \frac{a_{i-1}}{a_{m-i}} \leq \dots \leq \frac{a_{\lfloor \frac{m}{2} \rfloor - 1}}{a_{m-\lfloor \frac{m}{2} \rfloor}} \leq 1. \quad (1.3)$$

It is easily checked that the ratio monotonicity implies both log-concavity and the spiral property.

Let $P(x) = a_0 + a_1x + \dots + a_mx^m$ be a polynomial with nonnegative coefficients. We say that $P(x)$ is unimodal if the sequence $\{a_i\}_{0 \leq i \leq m}$ is unimodal. A mode of $\{a_i\}_{0 \leq i \leq m}$ is also called a mode of $P(x)$. Similarly, we say that $P(x)$ is log-concave or ratio monotone if the sequence $\{a_i\}_{0 \leq i \leq m}$ is log-concave or ratio monotone.

Throughout this paper $P(x)$ is assumed to be a polynomial with nonnegative and nondecreasing coefficients. Boros and Moll [2] proved that $P(x+1)$, as a polynomial of x , is unimodal. Alvarez et al. [1] showed that $P(x+n)$ is also unimodal for any positive integer n , and conjectured that $P(x+d)$ is unimodal for any $d > 0$. Wang and Yeh [6] confirmed this conjecture and studied the modes of $P(x+d)$. Llamas and Martínez-Bernal [5] obtained the log-concavity of $P(x+c)$ for $c \geq 1$. Chen, Yang and Zhou [4] showed that $P(x+1)$ is ratio monotone, which leads to an alternative proof of the ratio monotonicity of the Boros-Moll polynomials [3].

Let $M_*(P, d)$ and $M^*(P, d)$ denote the smallest and the greatest mode of $P(x+d)$ respectively. Our main result is the following theorem, which was conjectured by Wang and Yeh [6].

Theorem 1.1 *Suppose that $P(x)$ is a monic polynomial of degree $m \geq 1$ with nonnegative and nondecreasing coefficients. Then for $0 < d_1 < d_2$, we have $M_*(P, d_1) \geq M_*(P, d_2)$ and $M^*(P, d_1) \geq M^*(P, d_2)$.*

From now on, we further assume that $P(x)$ is monic, that is $a_m = 1$. For $0 \leq k \leq m$, let

$$b_k(x) = \sum_{j=k}^m \binom{j}{k} a_j x^{j-k}. \quad (1.4)$$

Therefore, $b_k(x)$ is of degree $m-k$ and $b_k(0) = a_k$. For $1 \leq k \leq m$, let

$$f_k(x) = b_{k-1}(x) - b_k(x), \quad (1.5)$$

which is of degree $m-k+1$. Let $f_k^{(n)}(x)$ denote the n -th derivative of $f_k(x)$.

Our proof of Theorem 1.1 relies on the fact that $f_k(x)$ has only one real zero on $(0, +\infty)$. In fact, the derivative $f_k^{(n)}(x)$ of order $n \leq m - k$ has the same property. We establish this property by induction on n .

2 Proof of Theorem 1.1

To prove Theorem 1.1, we need the following three lemmas.

Lemma 2.1 *For any $0 \leq k \leq m$, we have $b'_k(x) = (k+1)b_{k+1}(x)$.*

Proof. It can be checked that

$$\begin{aligned}
b'_k(x) &= \sum_{j=k}^m \binom{j}{k} a_j (x^{j-k})' \\
&= \sum_{j=k+1}^m (j-k) \binom{j}{k} a_j x^{j-k-1} \\
&= \sum_{j=k+1}^m (j-k) \frac{j!}{k!(j-k)!} a_j x^{j-(k+1)} \\
&= \sum_{j=k+1}^m \frac{j!}{k!(j-k-1)!} a_j x^{j-(k+1)} \\
&= \sum_{j=k+1}^m (k+1) \frac{j!}{(k+1)!(j-(k+1))!} a_j x^{j-(k+1)} \\
&= (k+1)b_{k+1}(x),
\end{aligned}$$

as required. ■

Lemma 2.2 *For $n \geq 1$ and $1 \leq k \leq m$, we have*

$$f_k^{(n)}(x) = (k+n-1)_n b_{k+n-1}(x) - (k+n)_n b_{k+n}(x), \quad (2.6)$$

where $(m)_j = m(m-1) \cdots (m-j+1)$.

Proof. Use induction on n . For $n = 1$, we have

$$f_k^{(1)}(x) = f'_k(x) = k b_k - (k+1) b_{k+1}.$$

Assume that the lemma holds for $n = j$, namely,

$$f_k^{(j)}(x) = (k+j-1)_j b_{k+j-1}(x) - (k+j)_j b_{k+j}(x).$$

Therefore,

$$\begin{aligned}
f_k^{(j+1)}(x) &= (k+j-1)_j b'_{k+j-1}(x) - (k+j)_j b'_{k+j}(x) \\
&= (k+j)(k+j-1)_j b_{k+j}(x) - (k+j+1)(k+j)_j b_{k+j+1}(x) \\
&= (k+j)_{j+1} b_{k+j}(x) - (k+j+1)_{j+1} b_{k+j+1}(x).
\end{aligned}$$

This completes the proof. ■

Lemma 2.3 *For $1 \leq k \leq m$ and $0 \leq n \leq m-k$, the polynomial $f_k^{(n)}(x)$ has only one real zero on the interval $(0, +\infty)$. In particular, $f_k(x)$ has only one real zero on the interval $(0, +\infty)$.*

Proof. Use induction on n from $m-k$ to 0. First, we consider the case $n = m-k$. Recall that

$$f_k(x) = \sum_{j=k-1}^m \binom{j}{k-1} a_j x^{j-k+1} - \sum_{j=k}^m \binom{j}{k} a_j x^{j-k}.$$

Thus $f_k(x)$ is a polynomial of degree $m-k+1$. Note that

$$f_k^{(m-k)}(x) = (m-k+1)! \binom{m}{k-1} a_m x + \left[\binom{m-1}{k-1} a_{m-1} - \binom{m}{k} a_m \right] (m-k)!.$$

Clearly, $f_k^{(m-k)}(x)$ has only one real zero x_0 on $(0, +\infty)$. So the lemma is true for $n = m-k$.

Suppose that the lemma holds for $n = j$, where $m-k \geq j \geq 1$. We proceed to show that $f_k^{(j-1)}(x)$ has only one real zero on $(0, +\infty)$. From the inductive hypothesis it follows that $f_k^{(j)}(x)$ has only one real zero on $(0, +\infty)$. In light of (2.6), it is easy to verify that $f_k^{(j)}(+\infty) > 0$ and

$$f_k^{(j)}(0) = (k+j-1)_j a_{k+j-1} - (k+j)_j a_{k+j} \leq 0.$$

It follows that the polynomial $f_k^{(j-1)}(x)$ is decreasing up to certain point and becomes increasing on the interval $(0, +\infty)$. Again by (2.6) we find $f_k^{(j-1)}(+\infty) > 0$ and

$$f_k^{(j-1)}(0) = (k+j-2)_{j-1} a_{k+j-2} - (k+j-1)_{j-1} a_{k+j-1} \leq 0.$$

So we conclude that $f_k^{(j-1)}(x)$ has only one real zero on $(0, +\infty)$. This completes the proof. ■

Proof of Theorem 1.1. In view of (1.4), we have

$$P(x+d) = \sum_{k=0}^m a_k (x+d)^k = \sum_{k=0}^m b_k(d) x^k.$$

Let us first prove that $M^*(P, d_1) \geq M^*(P, d_2)$. Suppose that $M^*(P, d_1) = k$. If $k = m$, then the inequality $M^*(P, d_1) \geq M^*(P, d_2)$ holds. For the case $0 \leq k < m$, it suffices to verify that $b_k(d_2) > b_{k+1}(d_2)$. By Lemma 2.2, $f_{k+1}(x)$ has only one real zero on $(0, +\infty)$. Note that

$$f_{k+1}(0) \leq 0 \quad \text{and} \quad f_{k+1}(+\infty) > 0.$$

From $M^*(P, d_1) = k$ it follows that $b_k(d_1) > b_{k+1}(d_1)$, that is $f_{k+1}(d_1) > 0$. Therefore, $f_{k+1}(d_2) > 0$, that is, $b_k(d_2) > b_{k+1}(d_2)$.

Similarly, it can be seen that $M_*(P, d_1) \geq M_*(P, d_2)$. Suppose that $M_*(P, d_2) = k$. If $k = 0$, then we have $M_*(P, d_1) \geq M_*(P, d_2)$. If $0 < k \leq m$, it is necessary to show that $b_{k-1}(d_1) < b_k(d_1)$. Again, by Lemma 2.2, we know that $f_k(x)$ has only one real zero on $(0, +\infty)$. From $M_*(P, d_2) = k$, it follows that $b_{k-1}(d_2) < b_k(d_2)$, that is $f_k(d_2) < 0$. By the boundary conditions

$$f_k(0) \leq 0 \quad \text{and} \quad f_k(+\infty) > 0,$$

we obtain $f_k(d_1) < 0$, that is $b_{k-1}(d_1) < b_k(d_1)$. This completes the proof. ■

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